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Decay of the envelope soliton under the action of δ-kicks and a 'soliton standard map'

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Abstract. The dynamics of an envelope soliton under the action of δ-kicks is investigated in the framework of a nonlinear Schrödinger equation. It is shown that soliton dynamics may be described by a 'standard map' for a sufficiently large number of δ-kicks. The estimation of the soliton lifetime and radiation level is derived by means of the inverse scattering transform.

1. Introduction

Progress in nonlinear dynamics is closely related to the analysis of a few universal or canonical models. These models must meet two primary requirements: they must describe a wide range of physical phenomena and allow for deep analytic investigation. An example of such a model describing a universal stochastic instability in Hamiltonian systems is a 'standard map' introduced by Chirikov (Chirikov and Zaslavsky 1971, and Lichtenberg and Lieberman 1983):

$$\bar{I} = I + Kf(\theta) \qquad \bar{\theta} = \Omega(\bar{I}) + \theta \tag{1}$$

where $f(\theta)$ is a 2π periodic function, K is a nonlinearity parameter, while I and θ have a sense of action-angle variables. This map can be obtained directly from an equation of a pendulum excited by δ-kicks:

$$\mathrm{d}^2x/\mathrm{d}t^2 + \sum_l \delta(t - l\tau)n(x) = 0. \tag{2}$$

A standard map makes a very good model: analytic results obtained for its stimulated numerous investigations concerned with both a deeper insight into the properties of dynamic chaos in Hamiltonian systems and generalization to other physical situations. A 'quantum analogue' (Chirikov *et al* 1981, Chirikov 1984) obtained by the quantizations of (2)

$$U_t = \mathrm{i}\left(U_{xx} + \sum_l \delta(t - l\tau)n(x)U \right) \tag{3}$$

is one of the most significant generalizations of the standard map.

The Schrödinger equation (3) is interesting as a description of a quantum system that behaves stochastically in a classical limit. The main problem here is the relation between classical and 'quantum' chaos. 'Quantum chaos' is the complex dynamics of narrow wave trains of the field $U(x)$. If the train width is much smaller than the variation of the potential $n(x)$, then by virtue of the WKB method for the mass centre of the wave train, we obtain again (2), which reduces to the analysis of the standard map (1)†. The train spreads with time, which results in the degeneration of 'quantum chaos' and in the onset of multifrequency oscillations that, naturally, cannot be interpreted within (1). Therefore the range of times for which (1) and (2) are applicable is one of the most significant problems in the theory of 'quantum chaos'. At present, some definite estimates of this value have been obtained (Chirikov and Shepelyansky 1986, Zaslavsky 1981, Berman and Kolovsky 1985).

On the other hand, since the wave train velocity is of the order of $\varepsilon = \|n(x)'\| \ll 1$, i.e. of the order of ε, the classical-like chaos must exist at times t_q greater than $1/\varepsilon$, otherwise, it would be meaningless. On the other hand, by virtue of the WKB approach, the range of times for which (2) is applicable must meet the condition $t_q \ll 1/\varepsilon^2$ (the wave train width becomes equal to $1/\varepsilon$ in this time). We thus obtain the applicability condition for a classical description: $1/\varepsilon < t_q \ll 1/\varepsilon^2$, which is quite acceptable when $\varepsilon \to 0$. It was shown in Chirikov (1984), Chirikov and Shepelyansky (1986) and Zaslavsky (1981) that by virtue of the exponential instability of the trajectories of map (1), the number of iterations (kicks) at which the train moves according to classical equations is, in fact, $\sim \ln(\varepsilon)$, i.e. $t_q \sim |\ln(\varepsilon)/\varepsilon|$.

There naturally arises a question: are there mechanisms to prevent the train from spreading and to stabilize the 'quantum chaos'. A positive answer was given in Aranson *et al* (1989): particle-like chaos exists at very large (greater than $1/\varepsilon^2$) times in a nonlinear analogue of (3):

$$U_t = \mathrm{i}(U_{xx} + n(x, t)U + |U|^2 U) \tag{4}$$

if the potential $n(x, t)$ is sufficiently smooth. The nonlinearity compensates the spreading in specified solutions—solitons of (4) thus stabilizing the 'quantum chaos'. Equation (4) describes, for example, the propagation of optical pulses in a smoothly inhomogeneous lightguide, etc. We can also 'speculate' that it is a generalization of (3) to the case of the 'nonlinear quantum theory of the field'.

Previous investigations (Aranson *et al* 1986, Aranson *et al* 1989) failed to solve the equation of the analytical estimates of the lifetime of solitons of (4). In the general case, this problem cannot be solved for an arbitrary potential $n(x, t)$. Numerical simulations (Aranson *et al* 1989) showed that this time is not smaller than $1/\varepsilon^2$. Fortunately, we can again refer to the generalization of the standard map (1) derived from the nonlinear analogue (3):

$$U_t = \mathrm{i}\left(U_{xx} + \sum_l \delta(t - l\tau) n(x) U + 2|U|^2 U \right). \tag{5}$$

From (5) we obtained the evolution equations for the soliton parameters and sufficiently rigorous estimates for the soliton lifetime. The soliton dynamics for the case under study is described by almost standard map equations and, therefore, can be stochastic. Below we will call it a 'soliton standard map' since it is related primarily to soliton parameters.

The paper is organized as follows. Parameter variation of a pure soliton state in one δ-kick will be determined by the inverse scattering transform (IST) in section 2. The spectrum of soliton radiation in one δ-kick will be found in section 3. Section 4

† Strictly speaking, chaos in (3) is impossible in view of the discreteness of the spectrum of eigenvalues of the quantum system given in the bounded region.

will give the estimate of the soliton lifetime and the number of iterations for which a standard map holds. Rigorous mathematical formulation and proof of all principal results make the problem valuable. Although our paper is 'physical' in form there is no doubt that rigorous mathematical substantiation is possible.

2. Evolution of soliton parameters

The map between successive δ-kicks will be derived from (5) using the procedure adopted by Chirikov and Shepelyansky (1986). Assuming that $U(x, t)$ is a continuous function at the instant of a δ-kick and integrating (5) over the time interval $l\tau - O \leq t \leq l\tau + O$, we obtain the following relation†:

$$U(x, t = l\tau + O) = U(x, l\tau - O)\exp(\mathrm{i}n(x)) \tag{6}$$

(below we will use the definition: $\bar{U}(x) \equiv U(x, t = l\tau + O)$). Apparently, the δ-kick only changes the phase of the wavefunction $U(x, t)$. The evolution of $U(x, t)$ between δ-kicks is described by a nonlinear Schrödinger equation (NSE):

$$U_t = \mathrm{i}(U_{xx} + 2|U|^2 U). \tag{7}$$

Equation (7) is a completely integrable system (Zakharov and Shabat 1971, Newell 1980) and can be solved using the Zakharov–Shabat IST. Consequentially, knowing $U(x, t)$ we can find $U(x, t + \tau)$ (it is possible to find an explicit solution of (7) only for a very special class of initial conditions; in the general case we will use the asymptotic technique). The problem of the existence of soliton is solved quite correctly: the soliton exists until the linear system:

$$\varkappa_{1x} + \mathrm{i}k\varkappa_1 = U\varkappa_2 \qquad \varkappa_{2x} - \mathrm{i}k\varkappa_2 = U^*\varkappa_1 \tag{8}$$

associated with (7) has a discrete eigenvalue.

Using IST one can proceed from the analysis of NSE to the investigation of algebraic equations for scattering data (some details will be discussed below), which simplifies the solution significantly and yields rigorous estimates.

Assume that the function $U(x, t)$ and the corresponding scattering data $S(k)$ before the lth δ-kick are known. At the instant of the δ-kick the scattering data $S(k)$ are transformed: $S_{l+1}(k) = F[S_l(k)]$ where F is a functional. The evolution of the scattering data between the δ-kicks is trivial (see, e.g., Newell 1980, Lamb 1980). Thus, the scattering transform may reduce the investigation of (6) to the analysis of a functional map through a τ-period. We failed to obtain this functional in the explicit form (a fundamental solution of system (8) with an arbitrary function $U(x, t)$ is needed for this purpose). Nevertheless, for a nearly soliton case it can be written in a very simple form.

For the construction of F we shall make use of some aspects of IST for NSE. The spectrum of the problem (8) for the localized function $U(x, t)$ is known to consist of a countable number of discrete eigenvalues k_n with the corresponding localized eigenfunctions of (8) and of a continuous spectrum filling the real axis of k with the corresponding eigenfunctions $\varkappa_{1,2}$ bounded in the infinity. The discrete eigenvalues correspond to solitons while the continuous spectrum corresponds to radiation. Below we shall be interested only in a single-soliton case, that is the only discrete eigenvalue of (8).

† The first and the last terms on the right-hand side of (5) can be neglected at the instance of a δ-kick.

For the analysis of (8) introduce a pair of so-called fundamental solutions having a different behaviour at infinity (Iost functions):

$$\varphi=\begin{pmatrix}\varphi_1(x,k)\\ \varphi_2(x,k)\end{pmatrix}=\begin{pmatrix}e^{-ikx}\\ 0\end{pmatrix}\qquad \text{when } x\to\infty$$
$$\psi=\begin{pmatrix}\psi_1(x,k)\\ \psi_2(x,k)\end{pmatrix}=\begin{pmatrix}0\\ e^{ikx}\end{pmatrix}\qquad \text{when } x\to\infty. \tag{9}$$

Besides (9), we can construct another pair of solutions (8):

$$\bar\varphi=\begin{pmatrix}\varphi_2^*(x,k^*)\\ \varphi_1^*(x,k^*)\end{pmatrix}\qquad \bar\psi=\begin{pmatrix}\psi_2^*(x,k^*)\\ -\psi_1^*(x,k^*)\end{pmatrix}. \tag{10}$$

The solutions (9) and (10) are linearly dependent and are related by

$$\varphi=c_{11}\psi-c_{12}\bar\psi\qquad \bar\varphi=c_{12}^*\bar\psi-c_{22}^*\psi$$
$$\psi=c_{21}\bar\varphi+c_{22}\varphi\qquad \bar\psi=-c_{21}^*\varphi+c_{22}^*\bar\varphi. \tag{11}$$

Besides, we can readily obtain the following expressions:

$$c_{12}(k)=-c_{21}(k)\qquad c_{11}(k)=c_{22}^*(k)\qquad |c_{11}(k)|^2+|c_{12}(k)|^2=1. \tag{12}$$

The coefficients $c_{11}(k)$ and $c_{12}(k)$ are known as scattering data. A continuous spectra in IST can, in fact, be described knowing only the value $R(k)=c_{11}(k)/c_{12}(k)$, which is interpreted as a reflection coefficient, and the information on the poles of the coefficient $c_{12}(k)$, which characterizes discrete eigenvalues.

For the derivation of the law of the variation of scattering data we shall have to solve two different problems: find the variation of the soliton prameters (related with a discrete eigenvalue) and of the parameters of a continuous spectrum. The first problem may be solved using a traditional perturbation theory for a soliton while the second one presents considerable difficulties and involves quite delicate investigations.

A single-soliton solution of (7) has a form (Lamb 1980)

$$U_0(x)=2\beta\exp(-i\theta)/\cosh(z) \tag{13}$$

where $z=2\beta(x-\xi)$, $\xi=-4\alpha t+x_0$, $\theta=\alpha/\beta z+\delta$, $\delta=2\alpha\xi+4(\alpha^2-\beta^2)+\delta_0$, β is the soliton amplitude, $v=-4\alpha$ is the velocity and x_0 and δ_0 are the constants describing the position of the mass centre of the soliton and its phase. The single soliton solution (13) has a corresponding non-refractive potential (i.e. $c_{11}(k)=c_{22}(k)=0$), with

$$c_{12}^0(k)=(k-k_0)/(k-k_0^*) \tag{14}$$

where

$$k_0=\alpha+i\beta \tag{15}$$

and unique non-degenerate eigenfunction

$$\varphi_0=\begin{pmatrix}\varphi_1^0\\ \varphi_2^0\end{pmatrix}=\tfrac{1}{2}\exp(-ik_0x)/\cosh(z)\begin{pmatrix}\exp(-z)\\ -\exp(-i\theta)\end{pmatrix}. \tag{16}$$

Assume that before the δ-kick we have an unperturbed soliton solution (13). Then, after the δ-kick the system (8) will take the form

$$\varphi_{1x}+ik\varphi_1=U^0 e^{in(x)}\varphi_2\qquad \varphi_{2x}-ik\varphi_2=-U^{0*}e^{-in(x)}\varphi_1. \tag{17}$$

In search of the soliton parameters after the δ-kick, (17) will be written as

$$\varphi_{1x}+\mathrm{i}k\varphi_1=(\bar{U}+\delta U)\varphi_2 \qquad \varphi_{2x}-\mathrm{i}k\varphi_2=-(\bar{U}^*+\delta U^*)\varphi_1 \tag{18}$$

where $\bar{U}$ is a new soliton solution and $\delta U = U^0\exp(\mathrm{i}n(x))-\bar{U}$. The solution to (18) will be sought in the form of a series

$$\varphi=\bar{\varphi}+\sum_{n=1}^{N}\varepsilon^n\varphi^{(n)}. \tag{19}$$

Substituting (19) into (18) for the function $\varphi^{(n)}$, we obtain a set of linear equations

$$\varphi_{1x}^{(n)}+\mathrm{i}k\varphi_1^{(n)}-U\varphi_2^{(n)}=\delta U\varphi_2^{(n-1)} \qquad \varphi_{2x}^{(n)}+\mathrm{i}k\varphi_2^{(n)}-U^*\varphi_1^{(n)}=\delta U^*\varphi_1^{(n-1)}. \tag{20}$$

The solvability condition of (20) is the orthogonality of the right-hand side to the localized eigenfunction of a conjugate problem:

$$\int_{-\infty}^{\infty}(\delta U\bar{\varphi}_2^2+\delta U^*\bar{\varphi}_1^2)\,\mathrm{d}x=0 \tag{21}$$

(here the fundamental solutions of $\varphi(x)$ correspond to the soliton state $\bar{U}(x)$). The orthogonality conditions may be fulfilled only for certain relations between the soliton parameters α, β and $\bar{\alpha}$, $\bar{\beta}$, before and after the δ-kick, respectively. In particular, we obtain the following expression for parameters α and β in the first approximation:

$$\bar{\beta}=\beta \qquad \bar{\alpha}=\alpha-n_x(x_0)/2. \tag{22}$$

Taking into account that $v=-4\alpha$ and that the soliton mass centre coordinates x_0 and x between the δ-kicks are related by a simple expression $x=x_0+\tau v$, we obtain the following return map:

$$\bar{v}=v+2n_x(x_0) \qquad \bar{x}=x_0+\tau\bar{v}. \tag{23}$$

In the simplest case $n(x)\equiv\cos(\gamma x)$, equation (23) is a well known standard map demonstrating strong stochastic properties (Chirikov and Zaslavsky 1971). Thus, the problem of the evolution of the soliton parameters reduces, in the first approximation, to the analysis of a such map, as in the case of system (2) or system (3). This circumstance justifies the term 'soliton standard map' used for (5). In spite of (3) the equations (23) describe correctly dynamics of (5) for a very large time interval.

3. Evolution of radiation

We shall make an attempt to solve the problem of the soliton radiation of waves under the action of δ-kicks. We shall first consider some physical aspects in order to estimate the level of radiation.

Note that $n_{xx}(x)$ rather than $n_x(x)$ is a small parameter for radiation. This fact can be proved by expanding $n(x)$ in the series $n(x)=n(x_0)+n_x(x_0)(x-x_0)+n_{xx}(x_0)(x-x_0)^2/2+\ldots$ in the neighbourhood of the soliton mass centre. If we neglect all the terms except the first two then the perturbation will not result in radiation: the perturbed state will be a pure soliton with the parameter α renormalized by virtue of (23). Therefore we shall take $n_{xx}(x_0)$ as a small parameter for radiation.

On the other hand, simple physical relations suggest that the radiation power must not depend on the sign of $n_{xx}(x_0)$, i.e. it must be proportional at least to $n_{xx}(x_0)^2$ (this estimate was given by Pikovsky (1985)). If it is proportional to $n_{xx}(x_0)$ or some other derivative of $n(x_0)$, the soliton state can, in principle, be amplified radiating quanta. This, however, contradicts the law of conservation of the 'total number of quanta' before and after the δ-kick:

$$N_q = \int |U(x, t)|^2 \, dx = \text{constant}. \tag{24}$$

Thus, it is natural to expect that the power $W(k)$ of the waves radiated by the soliton will be proportional to $n_{xx}(x)^2$, i.e. ε^4. Such a weak effect can be determined by a standard method of perturbations with respect to ε only up to the fourth-order approximation, which takes much time and effort. Therefore we shall employ below an asymptotic method that provides faster convergence. A characteristic feature of this method is the use of exact relations between the scattering data obtained from (8).

According to Lamb (1980), the radiation power $W(k)$ is

$$W(k) = -\ln|c_{12}(k)|^2/\pi. \tag{25}$$

We shall express the coefficients $c_{12}(k)$ and $c_{11}(k)$ through fundamental solutions (9). Assume that we know the fundamental solutions φ^0 and ψ^0 for $\delta U = 0$. The corresponding coefficients $c^0_{12}(k)$ and $c^0_{11}(k)$, by virtue of (11), are related to φ^0 and ψ^0 by trivial expressions

$$c^0_{11}(k) = \lim_{x\to\infty} \exp(-ikx)\varphi^0_2(x, k) \qquad c^0_{12}(k) = \lim_{x\to\infty} \exp(ikx)\varphi^0_1(x, k). \tag{26}$$

When $\delta U \neq 0$, the solutions φ and ψ can be represented in the form

$$\varphi = C_1(x)\varphi^0 + C_2(x)\psi^0 \tag{27}$$

where $C_1(x)$ and $C_2(x)$ are unknown functions. Substituting (27) into (20) and using (26) as well as $c^0_{12} = \varphi^0_1\psi^0_2 - \varphi^0_2\psi^0_1$ (Lamb 1980), we obtain the following exact integral relations between φ and ψ and $c_{11}(k)$ and $c_{12}(k)$:

$$c_{12}(k) = c^0_{12}(k) + \int_{-\infty}^{\infty} [\delta U\varphi_2\psi^0_2 + \delta U^*\varphi_1\psi^0_1] \, dx \tag{28}$$

$$c_{11}(k) = c^0_{11}(k)c_{12}(k)/c^0_{12}(k) - (1/c^0_{12}(k)) \int_{-\infty}^{\infty} [\delta U\varphi_2\varphi^0_2 + \delta U^*\varphi_1\varphi^0_1] \, dx. \tag{29}$$

Assume now that $\bar{U}$ and δU have the same sense as in the previous section, and calculate $|c_{12}(k)|^2$ $(c^0_{11}(k) = 0)$:

$$|c_{12}(k)|^2 = |1 - c^0_{11}(k)|^2 = \left|1 - \frac{1}{c^0_{12}(k)} \int_{-\infty}^{\infty} [\delta U\varphi_2\varphi^0_2 + \delta U^*\varphi_1\varphi^0_1] \, dx\right|^2. \tag{30}$$

Our calculations will need the functions φ^0 and ψ^0 in the form used in soliton solution for $\bar{U}$:

$$\begin{aligned} \varphi^0 &= \begin{pmatrix}\varphi^0_1\\ \varphi^0_2\end{pmatrix} = \frac{e^{-ikx}}{(k-\bar{k}^*_0)}\begin{pmatrix}k-\bar{\alpha}-i\bar{\beta}\tanh(z)\\ -i\bar{\beta}\, e^{i\theta}\cosh^{-1}(z)\end{pmatrix} \\ \psi^0 &= \begin{pmatrix}\psi^0_1\\ \psi^0_2\end{pmatrix} = \frac{e^{ikx}}{(k-\bar{k}^*_0)}\begin{pmatrix}-i\bar{\beta}\cosh^{-1}(z)\\ k-\bar{\alpha}+i\bar{\beta}\tanh(z)\end{pmatrix} \end{aligned} \tag{31}$$

where all parameters are the same as in (13) and (16). Correspondingly, we shall have

$$c^0_{12}(k) = (k-\bar{k}_0)/(k-\bar{k}^*_0) = (k-\bar{\alpha}-i\bar{\beta})/(k-\bar{\alpha}+i\bar{\beta}). \tag{32}$$

The functions $\varphi(x)$ and $\psi(x)$ will also be sought in the form

$$\begin{aligned}\varphi &= \varphi^0(x, k) + D_1\varphi^0(x, k) + D_2\psi^0(x, k)\\ \psi &= \psi^0(x, k) + D_3\psi^0(x, k) + D_4\varphi^0(x, k).\end{aligned} \tag{33}$$

The term φ in (30) can be replaced in the first approximation by φ^0, which yields:

$$|c_{12}(k)|^2 = \left|1 - \int_{-\infty}^{\infty} [\delta U(\varphi_2^0)^2 + \delta U^*(\varphi_1^0)^2]\, \mathrm{d}x\right|^2. \tag{34}$$

Substituting (31) into (34), and employing (25), we obtain

$$W(k) = \frac{1}{\pi}\int \frac{2\beta \exp[\mathrm{i}(kx-\theta)]}{(k-\bar{k}^*)^2 \cosh(z)} [\![\{\exp[\mathrm{i}n_{xx}(x_0)(x-x_0)^2/2] - 1\}(k - \bar{\alpha} + \mathrm{i}\beta \tanh(z))^2 - \beta^2\{\exp[-\mathrm{i}n_{xx}(x_0)(x-x_0)^2/2] - 1\} \cosh^{-2}(z)]\!]^2\, \mathrm{d}x. \tag{35}$$

In spite of the awkward integral in (35), the result for $W(k)$ is quite clear (see the appendix):

$$W(k) = \frac{\pi n_{xx}(x_0)^2}{16|k - \bar{k}_0^*|^4 \cosh^2[(k-\bar{\alpha})/2\beta\pi]}. \tag{36}$$

It is seen from (36) that the soliton radiation line is exponentially narrow and Doppler shifted due to the soliton motion. We thus verified that $W(k) \sim \varepsilon^4$, which confirms our intuitive physical supposition.

We shall also need an estimate for $c_{12}(k)$. By virtue of

$$\int [\delta U \varphi_2^0 \psi_2^0 + \delta U^* \varphi_1^0 \psi_1^0]\, \mathrm{d}x \sim \mathrm{O}(n_{xxx})$$

equation (28) yields

$$c_{12}(k) = (k - \bar{k}_0)/(k - \bar{k}_0^*) + \mathrm{O}(n_{xx}^2) \tag{37}$$

(the term $n_{xxx}(x_0)$ appearing in (28) can be compensated by a more exact fitting of $\bar{\alpha}$).

Our next step will be to take into account the response of radiation to the soliton motion. The simplest way is to use the exact expressions for the integrals of (7) in the scattering data representation (Faddeev 1980):

(a) 'total number of quanta'

$$N_q = \int |U(x)|^2\, \mathrm{d}x = \int W(k)\, \mathrm{d}k + 4\beta \tag{38}$$

(b) 'pulse'

$$P = \frac{1}{2\mathrm{i}}\int (U^* U_x - U_x^* U)\, \mathrm{d}x = \int 2kW(k)\, \mathrm{d}k - 8\alpha\beta. \tag{39}$$

The integrals (38) and (39) permit us to 'link' the radiation and the soliton reasonably.

Let us analyse the expression (38) for the 'total number of quanta'. Apparently, the NSE (5) does not change its form for the class of perturbations considered here. Hence, the soliton amplitude does not change in the absence of radiation $\beta = \bar{\beta} =$ constant. With the radiation taken into account the soliton is damped. Substituting (36) into (38), we obtain the law of radiation soliton damping

$$\bar{\beta} = \beta - \pi|n_{xx}(x_0)|^2 M_0/(64\beta^3) \tag{40}$$

where

$$M_0 = \int \cosh^{-2}(t/2\pi)/(1+t^2)\,\mathrm{d}t.$$

We can obtain an (exact!) expression for the law of pulse correction

$$P = -2\mathrm{i} \int (U^* U_x - U_x^* U)\,\mathrm{d}x = -4\bar{\alpha}\bar{\beta} + 2\int kW(k)\,\mathrm{d}k. \tag{41}$$

Because the initial state is a pure soliton, $P_0 = -8\alpha\beta$,

$$8\bar{\alpha}\bar{\beta} = 8\alpha\beta + 2\int kW(k)\,\mathrm{d}k - \int n_x |U^0|^2\,\mathrm{d}x. \tag{42}$$

Taking into account $v = -4\alpha$, we obtain

$$\bar{v} = v - \frac{1}{2\beta}\int kW(k)\,\mathrm{d}k + \frac{1}{2\beta}\int n_x|U^0|^2\,\mathrm{d}x + \frac{v}{2\beta}\int W(k)\,\mathrm{d}k. \tag{43}$$

Generally speaking, the integral $\int n_x|U^0|^2\,\mathrm{d}x$ in (43) can be calculated to any accuracy, but it is sufficient for us to have the first three terms:

$$\frac{1}{2\beta}\int n_x|U^0|^2\,\mathrm{d}x = 2n_x(x_0) + M_1 n_{xxx}(x_0)/(8\beta^2) + M_2 n_{xxxxx}(x_0)/(24\beta^4) \tag{44}$$

where

$$M_1 = \int_{-\infty}^{\infty} t^2\,\mathrm{d}t/\cosh^{-2}(t) = \pi^2/6 \qquad M_2 = \int_{-\infty}^{\infty} t^4\,\mathrm{d}t/\cosh^{-2}(t) = 7\pi^4/120.$$

In the final analysis we have a system of maps

$$\begin{aligned}
\bar{\beta} &= \beta - \pi|n_{xx}(x_0)|^2 M_0/(64\beta^3) \\
\bar{v} &= v + 2n_x(x_0) + M_1 n_{xxx}(x_0)/(8\beta^2) + M_2 n_{xxxxx}(x_0)/(24\beta^4) \\
&\qquad - \pi n_x(x_0)(n_{xx}(x_0))^2 M_0/(32\beta^4) \qquad (45)\\
\bar{x} &= x_0 + \tau\bar{v}.
\end{aligned}$$

One remarkable fact must be emphasized: (45) is a dissipative system—the soliton amplitude is damped with time. This does not contradict the conservatism of the initial system (5). Such a phenomenon has a simple explanation: the 'total number of quanta' persists only in a complete system, i.e. 'soliton + radiation'. Under the action of perturbations the soliton damps due to the emission of quanta. The quanta in an unbounded system go to infinity and cannot be reabsorbed by the soliton. This effect is completely equivalent to dissipation.

The dissipation was obtained in our theory only for the soliton amplitude β. As for the equation for velocity v, our taking radiation into account only affected the renormalization of the force acting on the soliton but did not give 'radiation viscosity'. This is explained by the fact that the numbers of quanta radiated in the x and $-x$ directions are equal in the first approximation. However, with subsequent corrections to $W(k)$ taken into account, the radiation will become non-isotropic, which will give radiation viscosity in the equation for v. Such a situation will be described by a dissipative analogue of the standard map (Vlasova and Zaslavsky 1983) in which a strange attractor exists.

4. Estimation of the soliton lifetime

In the previous section we obtained the transformation of the scattering data in one δ-kick. The next δ-kick will occur with a perturbed state: soliton + weak radiation. For a correct estimation of the soliton lifetime we shall have to take into account the effect of 'residual' radiation on the soliton dynamics.

We shall make use of the results obtained in section 3. Assume that before the lth δ-kick the field $U(x, t)$ can be represented in the form $U(x) = U_0(x) + \tilde{U}(x)$, where U_0 is the soliton part of the solution and $\tilde{U}$ is the radiation. Let the scattering data $c_{11}(k)$ and $c_{12}(k)$ of the form $c_{11}(k) \sim \mathrm{O}(n_{xx})$ and $c_{12}(k) = (k - k_0)/(k - k_0^*) + \mathrm{O}((n_{xx})^2)$ correspond to such a field. In search for $\bar{c}_{12}(k)$ and $\bar{c}_{11}(k)$ after the δ-kick, we shall write (8) in the form

$$\varphi_{1x} + \mathrm{i}k\varphi_1 - (U + \tilde{U})\varphi_2 = \delta U \varphi_2 \qquad \varphi_{2x} - \mathrm{i}k\varphi_2 + (U + \tilde{U})^* \varphi_1 = \delta U^* \varphi_1 \tag{46}$$

where $\delta U = U\,\mathrm{e}^{\mathrm{i}n(x)} - U' + \tilde{U}\,\mathrm{e}^{\mathrm{i}n(x)} - \tilde{U}'$, $\tilde{U}'(k, x) = \tilde{U}(k + \Delta\alpha, x)$, $\Delta\alpha = n_x(x_0)/2$ are taken in the sense (18). For the coefficient $c_{12}(k)$ we obtain the force (28) and (29):

$$|c_{12}(k)|^2 = \left| 1 - c_{11}^0(k, \tau)\frac{\bar{c}_{12}}{c_{12}^0} - \frac{1}{c_{12}^0}\int [\delta U(\varphi_2^0)^2 + \delta U^*(\varphi_1^0)^2]\,\mathrm{d}x \right|^2$$
$$\bar{k} = k + \Delta\alpha \qquad c_{12}(k)/c_{12}^0(k) = 1 + \mathrm{O}((n_{xx})^2). \tag{47}$$

The solution (31) can be taken as the functions φ^0 and ψ^0. Apparently

$$\left| \int [\delta U(\varphi_2^0)^2 + \delta U^*(\varphi_1^0)^2]\,\mathrm{d}x \sim \mathrm{O}((n_{xx})^2) \right|^2 = \mathrm{O}(\varepsilon^4). \tag{48}$$

Therefore $\bar{W}(k) \sim \varepsilon^4$, i.e. the order of magnitude does not change. This means that we can write an expression for the solution amplitude at the lth step:

$$\bar{\beta} = \beta - \mathrm{O}(n_{xx}(x_0)^2) = \beta - \mathrm{O}(\varepsilon^4). \tag{49}$$

The particular form of the correction will be considered below. Equation (49) allows for the estimation of the soliton lifetime: the number of iterations of (49) must not exceed ε^{-4}. Assuming the interval between the δ-kicks to be $\tau \sim 1/\varepsilon$, we obtain the soliton lifetime $T \sim \varepsilon^{-5}$. The soliton dynamics during this period is described to a high accuracy ($\mathrm{O}(\varepsilon^3)$) by a standard map of the form (23). Thus, the 'soliton chaos' lives anomalously long as compared with the 'linear quantum chaos'. This is related to a high level of soliton stability with respect to adiabatic perturbations.

Let us now calculate an explicit, as for a possible, form of (47). The integral in (47) is divided into two parts: one related to the soliton $\delta U_1 = U^0 \exp(\mathrm{i}n(x)) - U$ and the other to the radiation $\delta U_2 = \tilde{U}^0 \exp(\mathrm{i}n(x)) - \tilde{U}$.

The expression for the soliton part does not differ from (35) and (36). The second part describes the transformation of radiation under the action of the perturbation. This part of the integral can be calculated explicitly only assuming that the radiation 'follows' the soliton. Indeed, if the real size of the function $\tilde{U}(x)$ is much smaller than $1/\varepsilon$, then $\int (\tilde{U}^0 \exp(\mathrm{i}n(x)) - \tilde{U})\,\mathrm{d}x \sim \varepsilon^4$ and it can be neglected in comparison to the contribution of the soliton part δU_1. Then we shall have the following trivial relation:

$$|\bar{c}_{11}(k)|^2 = \left| c_{11}^0(k, \tau) - \frac{\mathrm{i}\pi n_{xx}(x_0)}{4(k - \bar{k}_0^*)^2 \cosh[(k - \bar{\alpha})/2\beta\pi]} \right|^2 \tag{50}$$

where by virtue of the law of the evolution of the scattering data we have

$$c_{11}^0(k, \tau) = c_{11}^0(k)\,\mathrm{e}^{\mathrm{i}k^2\tau}.$$

The expression (50) can be written as

$$|\bar{c}_{11}(k)|^2 = |c^0_{11}(k,\tau)|^2 - \frac{\pi^2|n_{xx}(x_0)|^2}{16(k-\bar{k}_0^*)^4\cosh^2[(k-\bar{\alpha})/2\beta\pi]}$$

$$-2\,\mathrm{Re}\left(c^0_{11}(k)\,\mathrm{e}^{\mathrm{i}k^2\tau}\frac{\mathrm{i}\pi n_{xx}(x_0)}{4(k-\bar{k}_0^*)^2\cosh[(k-\bar{\alpha})/2\beta\pi]}\right). \tag{51}$$

If (50) holds, we can readily verify that (45) have the same form for other δ-kicks†.

The approximation of the radiation 'following' the soliton is justified by the fact that the modes with a phase velocity close to a soliton velocity make the main contribution to the radiation. The perturbation 'forces' from the soliton a narrow wave train that moves, spreading slowly, together with the soliton. The train width is β at the initial moment of time. If the repetition interval of δ-kicks is about $1/\varepsilon$, then in view of the diffusion nature of (7) the wave train width by the next δ-kick will be $\beta/\varepsilon^{1/2} \ll 1/\varepsilon$ (the nonlinearity of the medium can be neglected). As long as the scale of the function is $\tilde{U}(x) \ll 1/\varepsilon$, the evolution of $\tilde{U}(x)$ can be determinated using the WKB approximation (Abdullaev and Zaslavsky 1983). We shall find that the velocity and the coordinates of the mass centre of the wave train vary also by virtue of the map (23).

The problem of the spreading of a narrow wave train was investigated in detail by Chirikov (1984), Chirikov and Shepelyansky (1986), Chirikov *et al* (1981) and Zaslavsky (1981). It was mentioned that the width of such a train can grow exponentially to the number of δ-kicks if the corresponding trajectory of the return map associated with (3) is stochastic. In view of the above, the 'standard soliton map' in the form (45) can be used in our analysis only for a finite and sufficiently small number of δ-kicks $N_0 \simeq -\ln|n_x|$. At larger times, the term related to the radiation in the integral (47) may become of the order of the soliton δU_1. Then, the form of the correction due to the radiation will change, although the order of magnitude will be retained. In the expression for $W(k)$ this is equivalent to the early history of the soliton motion.

5. Conclusion

We have investigated the problems of the dynamics and the lifetime of a single soliton in an NSE in the presence of weak perturbations. We have derived the return map for the soliton parameters, the level of 'radiation' and the laws of 'soliton-radiation' interaction. We have also obtained an unexpected result: the soliton lives for an extremely long time in the comparison with wave train within linear Schrödinger equation.

The technique proposed in our paper can be readily generalized to a multisoliton case. For this case we can obtain a system of coupled standard maps for the soliton velocities and the centre of mass coordinates. Because the radiation weakly influences the soliton dynamics at reasonable times, the equations describing a multisoliton system can be assumed to be finite dimensional in spite of continuous NSE. Moreover, the radiation makes the stochastic set in such a system attracting.

† When calculating the integrals with respect to k, the last term in (51) can be neglected because it contains, when $\tau \sim 1/\varepsilon \gg 1$, a multiplier $\exp(\mathrm{i}k^2\tau)$ rapidly oscillating with respect to k.

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Appendix

Denote the integral in (35) as I:

$$I = \int_{-\infty}^{\infty} [\delta U(\varphi_2^0)^2 + \delta U^*(\varphi_1^0)^2]\,\mathrm{d}x. \tag{A.1}$$

Take into account that

$$\begin{aligned}\delta U = U_0\,\mathrm{e}^{\mathrm{i}n(x)} - U &= [2\beta\,\mathrm{e}^{-\mathrm{i}\theta}/\cosh(z)]\{\exp[\mathrm{i}n_{xx}(x_0)(x-x_0)^2/2]-1\} \\ &= \mathrm{i}\beta n_{xx}(x_0)(x-x_0)^2\,\mathrm{e}^{-\mathrm{i}\bar{\theta}}/\cosh(z).\end{aligned} \tag{A.2}$$

Substituting into I expressions for φ_1^0 and φ_2^0 from (31), we obtain after obvious transformations the following equation:

$$\begin{aligned} I = \int_{-\infty}^{\infty} &\mathrm{i}n_{xx}(x_0)(x-x_0)^2\,\frac{\exp[\mathrm{i}(2kx-\theta)]}{(k-\bar{k}_0^*)^2} \\ &\times[2\beta/\cosh(z)(k-\bar{\alpha}+\mathrm{i}\beta\tanh(z))^2+\beta^3/\cosh^3(z)]\,\mathrm{d}x \\ &= \frac{-\mathrm{i}n_{xx}(x_0)}{8(k-\bar{k}_0^*)^2}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}\mu z)z^2}{\cosh(z)} \\ &\times[\mu^2-1+2\mathrm{i}\mu\tanh(z)+2/\cosh^2(z)]\,\mathrm{d}z \end{aligned} \tag{A.3}$$

where $\mu = (k-\alpha)/\beta$. Take into account that

$$I = \frac{-\mathrm{i}n_{xx}(x_0)}{8(k-\bar{k}^*)^2}\int_{-\infty}^{\infty}(\mathrm{e}^{\mathrm{i}\mu z})_{\mu\mu}[\mu^2-2\mathrm{i}\mu\partial/\partial z-(\partial^2/\partial z^2)]\cosh^{-1}(z)\,\mathrm{d}z. \tag{A.4}$$

Then after simple transformations, the integral I will take the form

$$I = \frac{\mathrm{i}n_{xx}(x_0)}{4(k-\bar{k}^*)^2}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}\mu z)}{\cosh(z)}\,\mathrm{d}z. \tag{A.5}$$

The integral (A.5) can now be calculated in terms of elementary functions:

$$\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}\mu z)}{\cosh(z)}\,\mathrm{d}z = \pi\cosh^{-2}(\mu/\pi) = \pi\cosh^{-2}[(k-\bar{\alpha})/(2\beta\pi)]. \tag{A.6}$$

Using (A.5) and (A.6) we finally obtain the expression (36).

References

Abdullaev S S and Zaslavsky G M 1983 *Zh. Eksp. Teor. Fiz.* **85** 1573
Aranson I S, Gorshkov K A and Rabinovich M I 1986 *Zh. Eksp. Teor. Fiz.* **86** 929; *Phys. Lett.* 1989 **139A** 65
Berman G P and Kolovsky I R 1985 *Physica* **17D** 183
Chirikov B V 1979 *Phys. Rep.* **52** 263; 1984 *Proc. Int. Conf. on Plasma Physics, Lausanne* vol 2 761

Chirikov B V, Izrailev F M and Shepelyasky D L 1981 *Sov. Sci. Rev.* **21** 209
Chirikov B V and Shepelyansky L D 1986 *Izv. Vuzov. Radiofizika* **29** 1041
Chirikov B V and Zaslavsky 1971 *Usp. Fiz. Nauk.* **106** 3
Faddeev L D 1980 Hamiltonian interpretation of inverse scattering transform technique *Solitons* ed R K Bullough and P I Caudrey (Berlin: Springer)
Lamb G L Jr 1980 *Elements of Soliton Theory* (New York: Wiley-Interscience)
Lichtenberg A I and Lieberman M A 1983 *Regular and Stoachastic Motion* (Berlin: Springer)
Newell A 1980 The inverse scattering transform *Solitons* ed R K Bullough and P I Caudrey (Berlin: Springer)
Pikovsky A S 1985 The breaking of solitons in chaotic motion. *Waves and Diffraction—85* (*Proc. All-Union Symposium on Diffraction and Propagation of Waves, Telavi, USSR*) (in Russian)
Vlasova O F and Zaslavsky G M 1983 *Phys. Lett.* **99A** 405
Zakharov V E and Shabat A B 1971 *Zh. Eksp. Teor. Fiz.* **61** 118
Zaslavsky G M 1981 *Phys. Rep.* **80** 157